Exercise 6A

1 a $\begin{pmatrix} 3 & -1 \\ 1 & 0 \\ 2 & 4 \end{pmatrix}$ dimension 3×2

b $\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$ dimension 2×2

c $\begin{pmatrix} 0 & -2 & 1 \\ 2 & 0 & -3 \\ -1 & 3 & 0 \end{pmatrix}$ dimension 3×3

d $\begin{pmatrix} 1 & 2 & 4 \end{pmatrix}$ dimension 1×3

2 a $\mathbf{A} = \begin{pmatrix} 2 & 4 \\ -3 & 6 \end{pmatrix}$

$\mathbf{A}^{\mathrm{T}} = \begin{pmatrix} 2 & -3 \\ 4 & 6 \end{pmatrix}$

b $$\mathbf{A}\mathbf{A}^{\mathrm{T}} = \begin{pmatrix} 2 & 4 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ 4 & 6 \end{pmatrix}$$
$$= \begin{pmatrix} 4+16 & -6+24 \\ -6+24 & 9+36 \end{pmatrix}$$
$$= \begin{pmatrix} 20 & 18 \\ 18 & 45 \end{pmatrix}$$

c $$\mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{pmatrix} 2 & -3 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ -3 & 6 \end{pmatrix}$$
$$= \begin{pmatrix} 4+9 & 8-18 \\ 8-18 & 16+36 \end{pmatrix}$$
$$= \begin{pmatrix} 13 & -10 \\ -10 & 52 \end{pmatrix}$$

3 a $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 6 \\ 0 & -4 \end{pmatrix}$

$$\mathbf{BA} = \begin{pmatrix} 1 & 6 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 3-12 & 2+6 \\ 0+8 & 0-4 \end{pmatrix} = \begin{pmatrix} -9 & 8 \\ 8 & -4 \end{pmatrix}$$

b $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix} \Rightarrow \mathbf{A}^{\mathrm{T}} = \begin{pmatrix} 3 & -2 \\ 2 & 1 \end{pmatrix}$

$\mathbf{B} = \begin{pmatrix} 1 & 6 \\ 0 & -4 \end{pmatrix} \Rightarrow \mathbf{B}^{\mathrm{T}} = \begin{pmatrix} 1 & 0 \\ 6 & -4 \end{pmatrix}$

$$\mathbf{A}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}} = \begin{pmatrix} 3 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 6 & -4 \end{pmatrix} = \begin{pmatrix} 3-12 & 0+8 \\ 2+6 & 0-4 \end{pmatrix} = \begin{pmatrix} -9 & 8 \\ 8 & -4 \end{pmatrix}$$

From part **a**:

$\mathbf{BA} = \begin{pmatrix} -9 & 8 \\ 8 & -4 \end{pmatrix} \Rightarrow (\mathbf{BA})^{\mathrm{T}} = \begin{pmatrix} -9 & 8 \\ 8 & -4 \end{pmatrix}$

Therefore:

$\mathbf{A}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}} = (\mathbf{BA})^{\mathrm{T}}$ as required

4 a $\mathbf{A} = \begin{pmatrix} 1 & -4 & 8 \\ 4 & -7 & -4 \\ 8 & 4 & 1 \end{pmatrix} \Rightarrow \mathbf{A}^{\mathrm{T}} = \begin{pmatrix} 1 & 4 & 8 \\ -4 & -7 & 4 \\ 8 & -4 & 1 \end{pmatrix}$

4 b $\mathbf{AA}^{\mathrm{T}} = \begin{pmatrix} 1 & -4 & 8 \\ 4 & -7 & -4 \\ 8 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 4 & 8 \\ -4 & -7 & 4 \\ 8 & -4 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1+16+64 & 4+28-32 & 8-16+8 \\ 4+28-32 & 16+49+16 & 32-28-4 \\ 8-16+8 & 32-28-4 & 64+16+1 \end{pmatrix}$$

$$= \begin{pmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{pmatrix}$$

$$= 81\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$= 81\mathbf{I}$ as required

5 a $\mathbf{A} = \begin{pmatrix} 0 & 3 & 5 \\ -3 & 0 & -1 \\ -5 & 1 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -4 & 1 & -1 \\ 1 & 5 & 2 \\ -3 & 0 & 3 \end{pmatrix}$

$\mathbf{C} = \mathbf{AB}$

$$= \begin{pmatrix} 0 & 3 & 5 \\ -3 & 0 & -1 \\ -5 & 1 & 0 \end{pmatrix}\begin{pmatrix} -4 & 1 & -1 \\ 1 & 5 & 2 \\ -3 & 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0+3-15 & 0+15+0 & 0+6+15 \\ 12+0+3 & -3+0+0 & 3+0-3 \\ 20+1+0 & -5+5+0 & 5+2+0 \end{pmatrix}$$

$$= \begin{pmatrix} -12 & 15 & 21 \\ 15 & -3 & 0 \\ 21 & 0 & 7 \end{pmatrix}$$

b $\mathbf{C} = \begin{pmatrix} -12 & 15 & 21 \\ 15 & -3 & 0 \\ 21 & 0 & 7 \end{pmatrix} \Rightarrow \mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -12 & 15 & 21 \\ 15 & -3 & 0 \\ 21 & 0 & 7 \end{pmatrix}$

Since $\mathbf{C} = \mathbf{C}^{\mathrm{T}}$, the matrix is symmetric.

6 a $\mathbf{A}=\begin{pmatrix}0&3&5\\2&0&-1\\1&1&0\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}1&1&-1\\0&1&0\\-1&0&3\end{pmatrix}$

$$\mathbf{AB}=\begin{pmatrix}0&3&5\\2&0&-1\\1&1&0\end{pmatrix}\begin{pmatrix}1&1&-1\\0&1&0\\-1&0&3\end{pmatrix}$$

$$=\begin{pmatrix}0+0-5&0+3+0&0+0+15\\2+0+1&2+0+0&-2+0-3\\1+0+0&1+1+0&-1+0+0\end{pmatrix}$$

$$=\begin{pmatrix}-5&3&15\\3&2&-5\\1&2&-1\end{pmatrix}$$

b $\mathbf{AB}=\begin{pmatrix}-5&3&15\\3&2&-5\\1&2&-1\end{pmatrix}\Rightarrow(\mathbf{AB})^{\mathrm{T}}=\begin{pmatrix}-5&3&1\\3&2&2\\15&-5&-1\end{pmatrix}$

$$\mathbf{A}=\begin{pmatrix}0&3&5\\2&0&-1\\1&1&0\end{pmatrix}\Rightarrow\mathbf{A}^{\mathrm{T}}=\begin{pmatrix}0&2&1\\3&0&1\\5&-1&0\end{pmatrix}$$

$$\mathbf{B}=\begin{pmatrix}1&1&-1\\0&1&0\\-1&0&3\end{pmatrix}\Rightarrow\mathbf{B}^{\mathrm{T}}=\begin{pmatrix}1&0&-1\\1&1&0\\-1&0&3\end{pmatrix}$$

Therefore:

$$\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}=\begin{pmatrix}1&0&-1\\1&1&0\\-1&0&3\end{pmatrix}\begin{pmatrix}0&2&1\\3&0&1\\5&-1&0\end{pmatrix}$$

$$=\begin{pmatrix}0+0-5&2+0+1&1+0+0\\0+3+0&2+0+0&1+1+0\\0+0+15&-2+0-3&-1+0+0\end{pmatrix}$$

$$=\begin{pmatrix}-5&3&1\\3&2&2\\15&-5&-1\end{pmatrix}$$

Hence:

$(\mathbf{AB})^{\mathrm{T}}=\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$ as required